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System with temporal-spatial noise

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This paper investigates systems driven by temporal-spatial noise using two models, i.e., a spatially periodic model, and a model with infinite globally coupled oscillators. The study shows that, for the first model, the temporal-spatial noise has a stronger effect on the transport of particles than the usual additive and multiplicative noise; for the second model, the temporal-spatial noise can restrict the appearance of the symmetry-breaking nonequilibrium phase transition, in contrast with the case when the system is driven by the usual multiplicative noise.

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When we study a stochastic system, we usually believe that the noise existing in the system is stochastic only with respect to time [1–5] (Ref. [2] corresponds to the phenomenon of stochastic resonance, Ref. [3] to non-equilibrium transition, Ref. [4] to the transport of particles induced by noise, and Ref. [5] to resonant activation). But practically, any matter inevitably contains a foreign substance and the space distribution of the foreign substance is wholly stochastic. In addition, when dealing with the problems of condensed matter systems, we always run into some stochastic variables (noise) with respect to space, such as the space distribution for the direction of electron spin, and so on. Because the foreign substance in the system is stochastic for space (or the space distribution for the direction of the electron spin is stochastic, and so on), the multiplicative noise caused by the external environmental fluctuation must be random for space and time (in general, we express the effect of the external environmental fluctuation on the system as multiplicative noise [6,7], and the influence of the internal thermal fluctuation on the system as thermal additive noise [6]), i.e., for the same time multiplicative noise is stochastic at different space. Above we indicate that there is multiplicative noise which is stochastic for time and space. There is also thermal additive noise which is stochastic for time and space. If temperature T of the system studied by us is stochastic with respect to space, thermal additive noise will be random for time and space (since the thermal additive noise strength is proportional to the temperature). To solve this kind of problem, we must use the noise that is stochastic with respect not only to time but also to space. There is another kind of temporal-spatial noise, which is stochastic only to time (not to space), such as the systems in which temperature T depends on space x, i.e., $T=T(x)$, and $T(x)$ is a deterministic function of x, and the systems containing inhomogeneous substance in which the space distribution of the substance is deterministic.

In general, one easily mistakenly believes that temporal-spatial noise can be converted to the usual multiplicative noise not related to space. But in practice, temporal-spatial noise cannot be transformed to noise not related to space, except for some special cases. It will be discussed below whether temporal-spatial noise can be converted to general noise not related to space. So, it becomes necessary to investigate the systems with temporal-spatial noise. In this paper I will first calculate the Fokker-Planck equation (FPE) for the systems driven by multiplicative temporal-spatial noise, and then study two models with this temporal-spatial noise.

Consider the following Langevin equations:

$$\dot{x}_i=f_i(\mathbf{x},t)+\sum_j g_{ij}(\mathbf{x},t)\xi_{ij}(\mathbf{x},t), \tag{1}$$

$$\langle\xi_{ij}(\mathbf{x},t)\rangle=0,\langle\xi_{ij}(\mathbf{x},t)\xi_{kl}(\mathbf{y},s)\rangle=2\delta_{jl}w_{ik}^{j}(\mathbf{x},\mathbf{y},t)\delta(t-s), \tag{2}$$

where $\mathbf{x}=\{x_1,x_2,\ldots,x_N\}$, $f_i(\mathbf{x},t)$ and $g_{ij}(\mathbf{x},t)$ are arbitrary deterministic functions of variables $\mathbf{x}$ and t, and $\xi_{ij}(\mathbf{x},t)$ is temporal-spatial Gaussian white noise. $x_i=x_i(t)$, $y_i=x_i(s)$, $\langle\,\rangle$ denotes averaging over temporal-spatial noise, and $w_{ik}^{j}(\mathbf{x},\mathbf{y},t)$ are deterministic functions of $\mathbf{x}$, $\mathbf{y}$ and t $[w_{ik}^{j}(\mathbf{x},\mathbf{y},t)>0]$. Now, we wish to get a FPE corresponding to Eqs. (1) and (2). To do this, we start with the Kramers-Moyal expansion coefficients [1,8]

$$D^{(n)}_{i_1i_2i_3\ldots i_n}(\mathbf{x},t)=\frac{1}{n!}\lim_{\tau\to 0}\frac{1}{\tau}\langle(\Delta x_{i_1}\Delta x_{i_2}\cdots\Delta x_{i_n}\rangle \quad (n\geqslant 1), \tag{3}$$

in which $i_1,i_2,i_3,\ldots,i_n=1,2,3,\ldots,N$ and

$$\Delta x_i=x_i(t+\tau)-x_i(t)$$
$$=\int_t^{t+\tau}f_i[\mathbf{x}(s),s]ds$$
$$+\sum_j\int_t^{t+\tau}g_{ij}[\mathbf{x}(s),s]\xi_{ij}[\mathbf{x}(s),s]ds.$$

Below we only study the Stratonovich case. The Ito case can be studied in the same way. We assume the expansion

$$f_i(\mathbf{y},s)=f_i(\mathbf{x},s)+\sum_l^N \partial_{x_l}f_i(\mathbf{x},s)[y_l-x_l]+\cdots,$$

$$g_{ij}(\mathbf{y},s)=g_{ij}(\mathbf{x},s)+\sum_l^N \partial_{x_l}g_{ij}(\mathbf{x},s)[y_l-x_l]+\cdots, \quad (4)$$

$$\xi_{ij}(\mathbf{y},s)=\xi_{ij}(\mathbf{x},s)+\sum_l^N \partial_{x_l}\xi_{ij}(\mathbf{x},s)[y_l-x_l]+\cdots. \quad (5)$$

Using Eqs. (4) and (5), we get

$$\begin{aligned}x_i(t+\tau)-x_i(t)&=\int_t^{t+\tau}f_i(\mathbf{x},s)ds+\int_t^{t+\tau}\sum_l^N[\partial_{x_l}f_i(\mathbf{x},s)][y_l-x_l]ds\\&+\cdots+\sum_j\int_t^{t+\tau}g_{ij}(\mathbf{x},s)\xi_{ij}(\mathbf{x},s)ds\\&+\sum_j\int_t^{t+\tau}g_{ij}(\mathbf{x},s)\sum_l^N[\partial_{x_l}\xi_{ij}(\mathbf{x},s)](y_l-x_l)ds\\&+\sum_j\int_t^{t+\tau}\xi_{ij}(\mathbf{x},s)\sum_l^N[\partial_{x_l}g_{ij}(\mathbf{x},s)][y_l-x_l]ds+\cdots.\end{aligned} \quad (6)$$

By iterating quantities $x_i(s)-x_i$ in Eq. (6) and taking the average with respect to the noise distributions, we have

$$\begin{aligned}\langle\Delta x_i\rangle&=\left\langle\int_t^{t+\tau}f_i(\mathbf{x},s)ds\right\rangle+\left\langle\int_t^{t+\tau}\sum_l^N[\partial_{x_l}f_i(\mathbf{x},s)]\int_t^s f_l(\mathbf{x},t')dt'ds\right\rangle\\&+\left\langle\int_t^{t+\tau}\sum_l^N[\partial_{x_l}f_i(x,s)]\sum_j\int_t^s g_{lj}(\mathbf{x},t')\xi_{lj}(\mathbf{x},t')dt'ds\right\rangle+\cdots+\left\langle\sum_j\int_t^{t+\tau}g_{ij}(\mathbf{x},s)\xi_{ij}(\mathbf{x},s)ds\right\rangle\\&+\left\langle\sum_j\int_t^{t+\tau}g_{ij}(\mathbf{x},s)\int_t^s dt'\sum_l^N[\partial_{x_l}\xi_{ij}(\mathbf{x},s)]f_l(\mathbf{x},t')ds\right\rangle\\&+\left\langle\sum_j\int_t^{t+\tau}ds\,g_{ij}(\mathbf{x},s)\sum_l^N[\partial_{x_l}\xi_{ij}(\mathbf{x},s)]\sum_m\int_t^s g_{lm}(\mathbf{x},t')\xi_{lm}(\mathbf{x},t')dt'\right\rangle+\cdots\\&+\left\langle\sum_j\int_t^{t+\tau}ds\,\xi_{ij}(\mathbf{x},s)\sum_l^N[\partial_{x_l}g_{ij}(\mathbf{x},s)]\int_t^s f_l(\mathbf{x},t')dt'\right\rangle\\&+\left\langle\sum_j\int_t^{t+\tau}ds\,\xi_{ij}(\mathbf{x},s)\sum_l^N[\partial_{x_l}g_{ij}(\mathbf{x},s)]\sum_m\int_t^s g_{lm}(\mathbf{x},t')\xi_{lm}(\mathbf{x},t')dt'\right\rangle+\cdots.\end{aligned} \quad (7)$$

Using Eqs. (3) and (7), in the limit $\tau\to0$, we get the drift coefficient

$$\begin{aligned}D_i^{(1)}(\mathbf{x},t)&=\lim_{\tau\to0}\frac{\langle\Delta x_i\rangle}{\tau}\\&=f_i(\mathbf{x},t)+\sum_l^N\sum_j\{(\partial_{x_l}w_{li}^j)g_{ij}(\mathbf{x},t)g_{lj}(\mathbf{x},t)\\&\quad+[\partial_{x_l}g_{ij}(\mathbf{x},t)]g_{lj}(\mathbf{x},t)w_{li}^j(\mathbf{x},\mathbf{x},t)\},\end{aligned} \quad (8)$$

in which $\partial_{x_l}w_{li}^j=\partial_{x_l}w_{li}^j(\mathbf{x},\mathbf{y},t)|_{\mathbf{y}=\mathbf{x}}$. Similarly, the diffusive coefficients read

$$D_{kl}^{(2)}(\mathbf{x},t)=\frac{1}{2}\lim_{\tau\to0}\frac{\langle\Delta x_k\Delta x_l\rangle}{\tau}=\sum_j g_{lj}(\mathbf{x},t)g_{kj}(\mathbf{x},t)w_{kl}^j(\mathbf{x},\mathbf{x},t), \quad (9)$$

and $D^{(n)}_{i_1i_2i_3\ldots i_n}(\mathbf{x},t)=0$, for $n\geqslant3$. Therefore, we obtain the FPE corresponding to Eqs. (1) and (2) $\partial_tP(\mathbf{x},t)=-\Sigma_i^N\partial_{x_i}D_i^{(1)}(\mathbf{x},t)P(\mathbf{x},t)+\Sigma_{kl}^N\partial_{x_k}\partial_{x_l}D_{kl}^{(2)}(\mathbf{x},t)P(\mathbf{x},t)$.

When $g_{ij}(\mathbf{x},t)=1$, the Fokker-Planck equation for Eqs. (1) and (2) has been derived by Klyatskin and Tatarskii [9]. But they did not consider the case when the system is driven by multiplicative temporal-spatial noise. In addition, the method used in this paper for the derivation of the FPE is different from that used by the authors in Ref. [9]. Here I used the method of the Kramers-Moyal expansion to derive the FPE, while Klyatskin and Tatarskii used the method of differentiating equation $P_t(\mathbf{x})=\langle\delta(\mathbf{x}-\mathbf{x}(t))\rangle$ directly (see Eq. (2.7) in Ref. [9]). When $g_{ij}(\mathbf{x},t)=1$, our result for the FPE is in accordance with that of Ref. [9] (see Eq. (2.10) of Ref. [9]).

From Eqs. (8) and (9), we can find that owing to the action of $w(\mathbf{x},\mathbf{x},t)$ and $\partial_{x_l}w^j_{li}(\mathbf{x},\mathbf{y},t)|_{\mathbf{y}=\mathbf{x}}$ on the stochastic system, a dramatic change in the characteristic features of the system may occur in contrast with the case when the temporal-spatial noise $\xi_{ij}(\mathbf{x},t)$ is replaced by the general noise $\xi_{ij}(t)$, except for the case when $w^j_{li}(\mathbf{x},\mathbf{y},t)=h^j_{li}(\mathbf{x}-\mathbf{y})$. When $w^j_{li}(\mathbf{x},\mathbf{y},t)=h^j_{li}(\mathbf{x}-\mathbf{y})$, from Eqs. (8) and (9) we have $D^{(1)}_i(\mathbf{x},t)=f_i(\mathbf{x},t)+\Sigma^N_l\Sigma_j([\partial_{x_l}h^j_{li}(0)]g_{ij}(\mathbf{x},t)g_{lj}(\mathbf{x},t)+h^j_{li}(0)g_{lj}(\mathbf{x},t)[\partial_{x_l}g_{ij}(\mathbf{x},t)])$ where $\partial_{x_l}h^j_{li}(0)=\partial_{x_l}h^j_{li}(\mathbf{x}-\mathbf{y})|_{\mathbf{x}=\mathbf{y}}$, and $D^{(2)}_{kl}(\mathbf{x},t)=\Sigma_j h^j_{kl}(0)g_{lj}(\mathbf{x},t)g_{kj}(\mathbf{x},t)$. Now, it is clear that temporal-spatial noise has the same influence on the stochastic system as the general noise [i.e., noise $\xi_{ij}(t)$]. For Gaussian white noise, the time factor of the correlation function is a δ function [10], i.e., $\delta(t-s)$ [see Eq. (2)]. One easily thinks of the case when the space factor of the correlation function is also a δ function, i.e., $w^j_{li}(\mathbf{x},\mathbf{y},t)=\Pi^N_m\delta(x_m-y_m)$. For this case, one can approximately take $\delta(x_m-y_m)$ as $(1/x^{(m)}_0)\exp[-(1/x^{(m)}_0)|x_m-y_m|]$, and choose $x^{(m)}_0$ appropriately according to the concrete problem studied by us. Then one gets $h^j_{li}(0)=\Pi^N_m 1/x^m_0$ and $\partial_{x_l}h^j_{li}(0)=(1/x^{(l)}_0)\Pi^N_m(-1)/x^{(m)}_0$.

There is one special case for which we can convert the temporal-spatial noise to the noise not related to space. If the temporal-spatial noise $\xi_{ij}(\mathbf{x},t)$ is only stochastic for t but not for $\mathbf{x}$, i.e., giving a certain time $t=t_0$, variable $\xi_{ij}(\mathbf{x},t_0)$ is not a stochastic function of $\mathbf{x}$ but a deterministic function of $\mathbf{x}$, so the temporal-spatial noise $\xi_{ij}(\mathbf{x},t)$ in Eq. (1) has the correlation function $\langle\xi_{ij}(\mathbf{x},t)\xi_{kl}(\mathbf{y},s)\rangle=2\sqrt{D_iD_k}\delta_{jl}u_{ij}(\mathbf{x})u_{kj}(\mathbf{y})\delta(t-s)$, i.e., $w^j_{ik}(\mathbf{x},\mathbf{y},t)=\sqrt{D_iD_k}u_{ij}(\mathbf{x})u_{kj}(\mathbf{y})$ [$u_{ij}(\mathbf{x})$ is a deterministic function of $\mathbf{x}$], and we can transform the temporal-spatial noise $\xi_{ij}(\mathbf{x},t)$ to the general noise by $\xi_{ij}(\mathbf{x},t)=u_{ij}(\mathbf{x})\xi_{ij}(t)$, where $\xi_{ij}(t)$ is a Gaussian white noise not related to space with $\langle\xi_{ij}(t)\rangle=0$ and $\langle\xi_{ij}(t)\xi_{kl}(s)\rangle=2\sqrt{D_iD_k}\delta_{jl}\delta(t-s)$. For this case, we can first get the corresponding Langevin equations with noise not related to space. Then from the Langevin equations, we can calculate the FPE. Otherwise, i.e., in the general case, temporal-spatial noise $\xi_{ij}(\mathbf{x},t)$ cannot be translated into the general noise not related to space.

Below we study two models driven by temporal-spatial noise.

(a) We first consider the transport of the overdamped Brownian particles in a spatially periodic stochastic system [4]. We assume temperature T of the system is not a constant. It is a stochastic variable with respect to x. So the thermal additive noise is temporally and spatially related (since the thermal additive noise strength is proportional to T). The Langevin equation of the particle is

$$\dot{x}=f(x)+\xi(x,t), \tag{10}$$

where $f(x)=-\partial_xU(x)$. $U(x)$ is a periodic function of x with period $L=b-a=2\pi$ ($a=0$ and $b=2\pi$). Here we take it as a symmetric function $\cos x$, and assume that the diffusion occurs on a circle with the period L. The additive temporal-spatial thermal noise is Gaussian white with zero mean and correlation $\langle\xi(x,t)\xi(y,s)\rangle=2Bkw(x)\delta(t-s)$, in which B is a constant friction coefficient, k is the Boltzmann's constant, and $w(x)$ is a correlation function for the stochastic variable $T(x)$ with respect to x. Here, we assume $w(x)=2-\cos(x/2)$. Of course, $w(x)$ can be an arbitrary function of variable x.

According to the above formulas (8) and (9) we can get the FPE for Eq. (10),

$$\partial_tP(x,t)=-\partial_xD^{(1)}(x)P(x,t)+\partial^2_xD^{(2)}(x)P(x,t), \tag{11}$$

where $D^{(1)}(x)=\sin x+(D/2)\sin(x/2)$, and $D^{(2)}(x)=D(2-\cos(x/2))$ ($D=Bk$). Now, the probability density $P(x,t)$ has the periodic boundary condition $P(a,t)=P(b,t)$.

Our interest for this problem is the stationary solution of the probability and its corresponding probability current. Using formula (11) in Ref. [11], we can obtain the stationary solution of Eq. (11),

$$P(x)=N\frac{\exp[\Phi(x)]}{D^2(x)}\int_a^b\exp[-\Phi(x')-\Phi(b)\theta(x-x')]dx', \tag{12}$$

where $\Phi(x)=\int_a^x[D^{(1)}(x)/D^{(2)}(x)]dx$, $\theta(x-x')$ is the Heaviside step function, and N the normalization constant. According to formula (8) in Ref. [11], the probability current reads

$$\begin{aligned}J&=N\{1-\exp[-\Phi(2\pi)]\}\\&=N(1-\exp\{-[(8+D)\ln 3-8]/D\}).\end{aligned} \tag{13}$$

Equation (13) shows that the condition under which the flux changes sign is that value $\Phi(2\pi)$ can vary from positive to negative or vice versa. There is a critical value D_0 of D. When $D>D_0$, the flux is positive; while when $D<D_0$, the flux is negative. The critical value D_0 can be obtained from $\Phi(2\pi)=0$. It is $D_0=8(1/\ln 3-1)$.

We wish to give some explanation of the origin of the nonzero probability current. Consider a solution $x(t)$ of Eq. (10) for a given realization of the noise. Then $-x(t)$ is also a solution of Eq. (10), with $\xi(x,t)$ replaced by $-\xi(-x,t)$. If $\xi(x,t)$ is uncorrelated with space or it is stochastic only to time (not to space), this solution $-x(t)$ would have the same probability as $x(t)$, and there would be no symmetric breaking. However, in the presence of noise correlation with space, the probability, even though it is Gaussian, does not have this symmetry. So a nonzero flux can appear due to symmetry breaking.

We note that Reimann *et al.* have considered the transport of mean field coupled oscillators in the case of symmetric potential [12]. In their paper, the asymmetry of the system is induced by a joint action of coupling and multiplicative noise [we also studied this phenomenon (see Ref. [13])]. But, in this paper, the asymmetry is produced by noise correlation with space.

If the spatially periodic system is only driven by usual thermal additive noise (the temperature is a constant), no transport can occur ($J=0$). Transport occurring with general thermal additive noise means that thermal fluctuation (only one heat source) is converted into work and implies a violation of the second law of thermodynamics. This is only for the case of the general additive noise not related to space. If the spatially periodic system is driven by spatially related thermal additive noise, as shown above, there is probably the transport. This does not violate the second law of thermodynamics. Now, we can believe that there are infinite heat sources for the system.

In addition, if the spatially periodic system is driven by the usual additive and multiplicative Gaussian white noise (there is no correlation between the additive and multiplicative noise) in the case of symmetric potential, no nonzero current can be induced since no symmetry breaking happens. Here I show that the additive temporal-spatial noise can produce nonzero net flux when the potential is symmetric. Thus, it is obvious that the temporal-spatial noise has stronger effects on the transport of particles for model (10) than the usual additive and multiplicative noises.

(b) Second we consider a model with N globally coupled oscillators driven by additive noise and multiplicative temporal-spatial noise. The multiplicative temporal-spatial noise are induced in the systems containing foreign substance (or other spatial stochastic factors) by the external environmental fluctuations. The stochastic differential equations for oscillators are

$$\dot{x}_i=f(x_i)+g(x_i)\xi_i(x_i,t)-(\epsilon/N)\sum_j^N (x_i-x_j)+\eta_i(t), \tag{14}$$

where $i=1,2,3,\ldots,N$. $f(x)$ and $g(x)$ are the continued nonlinear functions of x. $\xi_i(x,t)$ and $\eta_i(t)$ are Gaussian white noise with zero mean and correlations $\langle\eta_i(t)\eta_j(s)\rangle=2D\delta_{ij}\delta(t-s)$, $\langle\xi_i(x_i,t)\xi_j(y_j,s)\rangle=2D'\delta_{ij}w(x_i,y_j)\delta(t-s)$ $[D'w(x_i,y_j)>0]$, and $\langle\xi_i(x_i,t)\eta_j(s)\rangle=0$. ϵ is the globally coupling constant. The mean field $s=(1/N)\Sigma_i^N x_i$.

In the thermodynamic limit (i.e., $N\to\infty$), all the oscillators have an identical evolution given by the nonlinear stochastic equation

$$\dot{x}=f(x)+g(x)\xi(x,t)-\epsilon x+\epsilon s+\eta(t), \tag{15}$$

where $s(t)=\langle x(t)\rangle$, which represents the time-dependent order parameter. From formulas (8) and (9), the FPE for Eq. (15) can be got as follow:

$$\partial_t P(x,t)=-\partial_x D^{(1)}(x)P(x,t)+\partial_x^2 D^{(2)}(x)P(x,t), \tag{16}$$

in which $D^{(1)}(x)=f(x)-\epsilon x+\epsilon s+D'g'gw(x,x)+D'g^2w'(x,x)$, and $D^{(2)}(x)=D'g^2w(x,x)+D$. Under the natural boundary condition, the stationary solution of the FPE (16) is

$$P_{st}(x)=(1/N)\exp\left(\int^x dx\,\frac{f(x)-\epsilon x+\epsilon s-D'g(g'w-gw')-D'g^2\partial_x w}{D'g^2w(x,x)+D}\right), \tag{17}$$

where $w'=\partial w(y,x)/\partial y|_{y=x}$ and $\partial_x w=dw(x,x)/dx$. N is the normalization constant.

A more detailed analysis of Eq. (17) shows that so long as function $f(x)$ is odd, $g(x)$ odd or even, and $w(x,x)$ even, it can be observed from Eq. (17) that any states $\{x_i(t)\}$ are identical with states $\{-x_i(t)\}$. So a symmetry-breaking nonequilibrium phase transition will appear [3]. The mean field is $s=\langle x\rangle=F(s)=\int_{-\infty}^{\infty}xP_{st}(x)dx$, from which the order parameter can be obtained by $m=|s|$. The critical condition for the nonequilibrium phase transition is $\partial_s F(s)|_{s=0}=1$, i.e., $F'(s=0)=1$ [3]. If the order parameter of the nonequilibrium phase transition changes continuously, the nonequilibrium phase transition will be of second order, and possess features similar to those observed at the second-order equilibrium phase transition. If the parameter of the nonequilibrium phase transition changes discontinuously, the nonequilibrium phase transition will be of first order and will have features similar to those at the first-order equilibrium phase transition.

If noise $\xi_i(x_i,t)$ in model (14) is not temporal-spatial, but only general, i.e., $\xi_i(t)$, the condition for the appearance of a symmetry-breaking nonequilibrium phase transition is that $f(x)$ is odd, and $g(x)$ is odd or even. But, as the above analysis, the condition for the emergence of a symmetry-breaking phase transition for model (14) is that $f(x)$ is odd, $g(x)$ odd or even, and $w(x,x)$ even. Thus, we can say that temporal-spatial noise can restrict the appearance of the symmetry-breaking nonequilibrium phase transition for model (14).

Temporal-spatial noise always exist in a lot of scientific fields, including biology, physics, chemistry, and their overlapping sciences. My result for the study of systems with the temporal-spatial noise can provide a theoretical foundation for studying further these systems.

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